THE
MATHEMATICAL GAZETTE.

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.; Prof. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.
Prof. E. T. WHITTAKER, M.A., F.R.S.; W. E. HARTLEY, B.A.

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ON HIGHER TRIGONOMETRY.

An article by Mr. G. H. Hardy in the March number of *The Mathematical Gazette* (coupled with certain other articles in recent numbers) raises several points which can hardly be allowed to pass without further discussion. It is proposed to discuss these points under three different heads, of which the first two are treated here while the third will find space in a later number of the *Gazette*.

I.—THE "HIGHER" IN MATHEMATICS.

Mr. Hardy begins by asking the question: "How are we to define what we mean by 'Higher Trigonometry'?" and his answer is a somewhat superficial *description*, not the definition of a selective principle.

Might one be permitted to suggest that it is the notion of "a limit" that provides the dividing-line above which the 'Higher' Mathematics (which used only to be mentioned with bated breath by those outside the charmed circle!) appears; Calculus on the one hand, Higher Trigonometry on the other pointing the upward path to the cloudy regions of modern analysis? [Regarding these two fundamental branches of Higher Mathematics, the writer would like at once to register his opinion that the former is much better suited to a school-course, and that the latter "most fascinating and instructive region of mathematics" (Mr. Hardy, p. 284) provides meat essentially for strong men and not for babes. To take a personal illustration, the writer himself, at the age of 14 years, knowing little algebra, less trigonometry and absolutely nothing of infinite series, came under the tuition of a master (certainly a man of very

striking character) who took up the Differential Calculus with a class of 20 boys (average age perhaps 16), and by means of it gave him an interest in all parts of the subject which culminated in a permanent connection with mathematics. The experiment was altogether a great success, and some such course, under a different master, sends to our university from that school the youths best equipped for mathematical work.]

Starting then with the above-mentioned principle of separation, some account of the logical evolution of the limit-conception does not seem out of place, especially since such an account is not easily to be found; even in the 2nd edition of Chrystal's Part II. the matter is scattered and to some extent incomplete.

The term *limiting value* is originally introduced on account of the fact that "a function of x" may have a value for a given value (a) of the argument, although "the expression," $f(x)$, fails to provide a value when a is substituted for x in it; this is because the function is regarded as having an identity of its own, apart from any particular expression of it (a fact which is associated with the possibility of expressing it in a variety of ways).

An expression $f(x)$ is the symbolical embodiment of instructions by the aid of which the value of the function corresponding to a given value of the argument, may be obtained; but cases arise in which the processes indicated cannot be carried out, *e.g.* the expressions $1/(x-a)$, $\sin(x-a)/(x-a)$ do not provide a value when x has the value a. In such a case it may be that another expression, algebraically equivalent to $f(x)$ when both give values, gives a value when x has the value a, *e.g.* the value $+1$ of the function expressed by $\sin(x-a)/(x-a)$ may be obtained from the expression $\{1-\theta(x).(x-a)^2/3!\}$ in which $\theta(x)$ has positive proper fractional values for all real values of x. In all such cases it may be said that the expression $f(x)$ establishes a *continuum of values* which belong to the function, and that there is one number (l) which, if it be regarded as corresponding to the value a of the argument, will *complete* this continuum: $f(x)$ is then said to have l as a limiting value for the value a of x, or to "tend to l as x tends to a"; and the analytical test of continuity for the case of a real function, demands that: corresponding to any assigned finite positive real number (ϵ), however small, there must be a finite positive real number (h) such that $|f(a+\theta h)-l|<\epsilon$, θ being a variable whose values lie between -1 and 0, between 0 and $+1$, or between -1 and $+1$. This test does not provide information as to the *existence* of a limiting value; and the comparative simplicity of the Differential Calculus depends on the fact that in the limit-problems raised by it, the limits where they occur can be actually evaluated, *i.e.* the one number which sustains the test can be obtained by elementary algebraic methods.

The notion of *limiting value* is, then, primarily associated with the values of a *dependent* variable; but the logical extension of the conception demanded by one particular type of function provides the useful symbols $\pm\infty$ which *can only be regarded as limiting values*, and yet may be given as values to an *independent* variable. Simple examples will indicate more clearly what has necessarily to be stated somewhat vaguely in its general aspect:

Consider (i) the function determined by the two equivalent expressions $(x^2-1)/(x-1)$ and $(x+1)$. With reference to the former expression the value $+2$ (obtained from the latter) corresponding to the value $+1$ of x, is said to be a limiting value; the peculiarity is, however, in no way associated with the abstracted symbols $+1$, $+2$, but resides wholly in the mode of expressing the function under consideration. If, on the other hand, (ii) $f(x)=1/(1-x)$ we write $f(1-0)=+\infty$, $f(1+0)=-\infty$, meaning that the continuum of functional values determined by values of x less than $+1$ is such that no positive real number can be chosen so great that it does not belong to this continuum, etc.; it is then said that "$f(x)$ tends to infinity as x tends to $+1$." *But no matter which of its expressions is used, the function cannot be regarded as having, when $x=a$, a* **value** *in the original sense,*

i.e. $\pm\infty$ are *essentially limiting values.*

We are now logically compelled to admit $\pm\infty$ to positions among the symbols belonging to the real number continuum and to consider the possibilities that arise from permitting $\pm\infty$ to be given as (limiting) values to a function-argument. [In the Dedekind-scheme $\pm\infty$ will have *schnitts* in which all the rationals are relegated to one class, leaving none for the other.]

The formula $f(\infty)=l$, like $f(a)=\infty$, represents essentially a limit-statement—the burden being in this case thrown on the *independent* variable; and this statement is expressed in finite terms as follows: corresponding to any assigned finite positive real number (ϵ), however small, a finite, positive real number (p) can be found such that $|f(x)-l|<\epsilon$ for all values of x greater than p.

There is, however, this other peculiarity about the tending of the argument to either of the limiting values possible for it, that the corresponding functional variation cannot be subjected to anything like the same closeness of scrutiny as in any finite range of argument-values; and although the case in which x tends to infinity by continuous variation can be (and generally is), reduced to the ordinary case by a "substitution," *e.g.* $x'=1/(x-a)$, a further extension of the limit-conception clearly demands consideration, to cover the possibility of a function having what

may without violence be called "a limiting value" when its argument tends to infinity by a system of values which do not form a continuum: in particular to meet the important case in which the expression $f(n)$ has only a meaning for *integral* values of n, *e.g.* if $f(n)=1-\frac{1}{2}+\frac{1}{3}\ldots+1/(2n-1)$.

The notion of limiting value is thus in this one ultimate case separated from its dependence on continuity*; and this separation marks broadly the dividing-line between the two fundamental branches (Differential Calculus and Higher Trigonometry) of Higher Mathematics.

The distinguishing feature of Higher Trigonometry is its concern with "infinite sequences," *i.e.* with the variation of functions when the argument tends to infinity by *integral* variation; and the essential difficulty of this branch of mathematics is to be found in the fact that, in general, the infinity of the argument cannot with advantage be got rid of by a substitution such as $x'=1/(x-a)$, and that, in consequence, algebraic transformations of the functional expression do not provide a simple means of determining the limiting value when it exists (and so of establishing its existence). Such cases as that of an infinite series of terms in geometric progression are quite exceptional, on account of the fact that the finite series can be "summed"; the treatment of the infinite series thus having the same kind of simplicity as the demonstration of $D.x^m=mx^{m-1}$, because a "closed" expression can be found which is equivalent to the "unclosed" expression $f(n)$; it is, therefore, suitable for inclusion in a school-course. But this does not, in the writer's opinion, provide a satisfactory starting-point for the theory of infinite series; and the summing to infinity of the geometric series can only be regarded as having an interest in connection with Recurring Decimals, until its importance to that theory is made clear.

Some method then has to be taken for the purpose of introducing the student gently to the theory of Infinite Series; and it would appear that this is most suitably to be found through the Circular and Exponential functions. If this be granted, the method (which has found much favour since it was invented for the Binomial Theorem by Euler) advocated by Mr. Hardy [cf. (B), that $\left(1+\frac{1}{1!}+\frac{1}{2!}+\ldots\right)^x=\left(1+x+\frac{x^2}{2!}+\ldots\right)$] must be considered "illogical"—besides that it is perhaps open to a charge of "artificiality"; these, however, are hard words, which may not be used with confidence until the mathematical circle has been

* The reason for this somewhat astonishing dissociation of two fundamentally interrelated conceptions is perhaps to be found in the fact that, in the very nature of things, mathematics can only deal with continuity by discrete approximation.

rounded off. Just to take one point, the use of this method leads Mr. Hardy to the early declaration that "*irrational* powers should at this stage be severely and *explicitly* left alone"—yet the fact that it satisfies $D.y=y$ is its simplest fundamental property! Indeed this equation provides perhaps the most "natural" way of introducing the function; and, apart from the variety of other criticisms possible, *it* does not throw into relief the exponential characteristic.

It remains to develop in a rigorous, and at the same time satisfactory, way the old process—which Mr. Hardy repudiates; taking the function $(1+z/n)^n$ as in itself an interesting case for binomial expansion, dealing only with integral values of n so as to avoid worse things, and obtaining thus the infinite series in a straightforward way which exhibits its exponential relations. The objections to such a process really hinge round the difficulty that is found in establishing, at this stage of the student's course, theorems of existence for limiting values, and this difficulty has now to be discussed.

II.—LIMIT OF A CONVERGENT SEQUENCE.

If the function $f(n)$ tends to a limit as n tends to infinity by positive integral variation, it is quite simply shown that corresponding to any assigned ϵ, a value (N_ϵ) of n can be obtained such that $|f(N_\epsilon+p)-f(N_\epsilon)|<\epsilon$ for all positive integral values of p.

The converse, viz., that if corresponding to any assigned finite positive real number (ϵ), however small, a finite value (N_ϵ) of n can be determined such that $|f(N_\epsilon+p)-fN_\epsilon|<\epsilon$, $f(n)$ tends to a limit, *i.e. the criterion for the existence of a limit to the sequence* is by no means so obvious; and one need have little hesitation in asserting that its abstruseness of demonstration has been the primary cause of the obscurity in which the fundamentals of Higher Trigonometry have remained: witness recent articles in the *Gazette*.

It is interesting to note that in the first edition of Chrystal's *Algebra* [chap. xxvi., § 3] the proof given for the sufficiency of this criterion is only a proof of the fact that *there cannot be two different limiting values*, and that in a work of so thorough a character the second edition merely provides an apologetic footnote confessing to want of rigour—not making the necessary changes, but giving a reference to the section on number-theory which has been included in that edition! This is in itself sufficient evidence of incongruity in the means employed to obtain rigour in so fundamental a thorem. Hobson follows Chrystal in this particular laxity [*Trigonometry*, § 193, chap. xiv.]. Whittaker, who draws particular attention to the importance of the theorem [*Analysis*, § 6] has a proof which is

heavily encumbered with algebraic detail and which only in the end succeeds in reducing the general theorem to what the writer of these remarks regards as merely a more intuitively obvious case [see Theorem III. below]. Dedekind himself, without whose number-theory no simple treatment of higher mathematics seems possible, made a courageous attempt to apply his theory directly to the problem under consideration,* and his demonstration has been reproduced in various places;† but the argument is not convincing, and the present writer at least is forced to the conclusion that the fact has been implicitly used in the proof of it. It is thus particularly gratifying that this most persistent obstacle has been removed in the pages of the *Gazette* [Oct. 1905, p. 236] by Professor E. B. Elliott, who undoubtedly went right to the heart of the matter when he applied the method of proof that has been, at least since the days of Euclid, associated with converse-theorems; his double application of the *Reductio ad Absurdum* cannot fail to strike everyone with admiration. One of his readers, however, disputes the validity of his preliminary theorem (1) [*Gazette,* May 1906, p. 327]; but Mr. Jourdain himself provides the explanation of his own error in the implication that a certain order of ideas is obligatory, when he states that "logically a definition of a real number (*i.e. a sequence* $s_1, s_2, \ldots$ *of rationals such that, where* ϵ *is any positive rational, there is an integer* n, *which depends on* ϵ, *such that for every* p $|s_n - s_{n+p}| < \epsilon$) precedes the criterion for the existence of a limit;" the italics have been supplied to indicate the abstruseness of the number-theory which has prevented the possibility of a sound elementary treatment of higher mathematics. All that need be said more is that Dedekind has waited long for the appreciation of his discovery of the arithmetical "essence of continuity," which made Nov. 24, 1858, the red-letter day of his life [cf. Essay I., Introduction]! The specification of a real number requires only a *schnitt* of the rationals, not a convergent sequence; the continuum of real numbers comes first, according to the views expressed in Section I. of this paper, and convergency is a conception derived from that of continuity.

The following presentation of the limit-theorems is based on Professor Elliott's theorem (1), the enunciation and demonstration of which given here have his approval:

I. *If the function* f(n) *does not tend to a limit as* n *tends to infinity, a finite positive real number* (κ) *can be determined such that whatever value* z *has, and whatever positive integral value* n *has, there is at least one positive integral value of* P_n *for which* $|f(n+P_n)-z| > \kappa$.

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* *Essays on Number*, by Richard Dedekind (Open Court Publishing Co., Chicago), I. vii., p. 24.

† Recently by Fine, *College Algebra* (Ginn & Co.), § 197, p. 61.

Proof. If this is not so, then there must be at least one value (l) of z, and a positive integer (N), such that for *all* positive integral values of p $|f(N+p)-l| < \epsilon$, *however small a finite positive real value be given to* ϵ;

i.e. $f(n)$ tends to the limiting value l, which contradicts the hypothesis.

II. *If* f(n) *is a real function which increases (decreases) continually as* n *tends to infinity, but remains always less (greater) than a certain specified real number* (a), f(n) *tends to a limiting value which* $\not>$ ($\not<$) a.

Proof. Confining our attention to the case of an increasing function, let us suppose that there is no limiting value; then by Theorem I., a finite positive real number (κ) can be found such that for this simple case $f(n) < a-\kappa$ for all values of n; hence, by a second application of the theorem, $f(n) < a-2\kappa$ for all values of n, and so on; *i.e.* $f(n)$ must be a continually increasing function which remains always less than any assigned real number—which is absurd.

III. *If* f(n), F(n) *are two real functions such that* f(n) *increases continually and* F(n) *decreases as* n *tends to infinity, if* F(n) *has no value which is less than any value of* f(n), *and if* F(n)−f(n) *can be made as small as we please by choosing* n *sufficiently great, the two sequences determine a single real number which is the limiting value of each function.*

Proof. If $a=F(a)$ where a denotes a finite value of n, the function $f(n)$ satisfies the requirements of Theorem II.; hence $f(n)$ tends to a limit (l), and similarly $F(n)$ tends to a limit (L). Suppose, if possible, that $(L-l)$ is finite, say δ (clearly positive); by hypothesis we can choose a value (N) of n such that $F(N)-f(N) < \delta$; but $F(N) > L$ and $f(N) < l$—which is absurd.

Finally, and most generally,

IV. *If corresponding to any assigned finite positive real number* ϵ, *however small, a finite positive integer* (N_ϵ) *can be found, such that* $|f(m)-f(n)| < \epsilon$ *provided only that* m, n *have integral values greater than* N_ϵ; f(n) *tends to a limit* [cf. Professor Elliott's Theorem (2)].

Proof. If there is no limit, then (by Theorem I.) a finite positive real number (κ) can be found such that

$$|f(n+P_n)-f(n)| > \kappa,$$

there being at least one positive integral value of P_n for each value of n. But, by hypothesis, if $n > N_\kappa$, $|f(n+p)-f(n)| < \kappa$ for *all* positive integral values of p. These inequalities being contradictory, it follows that $f(n)$ must tend to a limiting value.

These theorems form a sound elementary basis for the work of the higher analysis, being fundamental to the theory of Infinite Series and to the Integral Calculus. D. K. PICKEN.

MATHEMATICS FOR ARMY CANDIDATES.

DURING the last few years the methods of teaching Mathematics in schools have been subjected to much adverse criticism: this has induced teachers to apply themselves to the task of setting their house in order. It is now generally deemed advisable to give greater prominence to practical applications, to make more use of accurate drawing as a method of solution of problems and for purposes of verification, to curtail numerical calculations by dispensing with figures of doubtful or unnecessary accuracy, and to lay more stress on the employment of graphical methods.

These improvements have been advocated by the Committee of the Mathematical Association, and few will be disposed to deny that, apart from any advantages of the much vaunted useful order, such changes render the subject more attractive to the average boy than was the case when the connection between theory and practice was not always clearly shown.

If these reforms were carried out, as they were so intended, by the teachers themselves, and due attention were given to the points which in the past may have been unduly neglected, the educational efficacy of Mathematics would no doubt be increased, and the pupil would not only obtain a better grasp of the theory, but would be better able to apply it in practice. But behind the schoolmaster is the examiner; and, unfortunately, the views of some educational authorities by no means coincide with the programme of reforms propounded by those who wished to see the teaching carried out on more practical lines.

The policy of "overturn, overturn" has been pushed too far: if numerical calculations, graphical representations, and geometrical drawing are not merely to be an aid, but a substitute for a systematic knowledge of the theory of Elementary Mathematics, the value of the subject as an educational instrument will be but small.

To show that there is a real danger of such a revolution, we wish to draw attention to the class of knowledge which is now demanded of candidates for commissions in the army.

There may be different opinions as to the educational value of a subject taught according to any given syllabus, but every one will agree that, when examining bodies demand knowledge of a special kind, their requirements should be stated as carefully and precisely as possible. That the knowledge required by the Civil Service Commissioners is of a very special kind will be readily conceded, but the language in which their wants are formulated leaves much to be desired, though the statements of what they do not require sometimes form oases of the blest and the intelligible.

Under the heading of Arithmetic, for the Qualifying Examination, we find "mensuration of plane figures and solids." Surely a departure of this magnitude should be defined a little more carefully. The Algebra is "to simple quadratic equations"; but "skill in elaborate analysis will not be looked for." In the list of subjects for the Competitive Examination the Trigonometry required is "up to and including solution of plane triangles"; but nothing is said as to formulae involving two angles or identities, and we are left to derive what comfort we may from the usual footnote disclaiming any demand for "great analytical skill." The schedule of Geometry issued by the University of Cambridge is used to define the Geometry; this, so far as the syllabus is concerned, is satisfactory, and we wish that similar means had been employed in other cases. The Algebra for Math. II. includes "a working knowledge (without rigorous fundamental demonstration) of the elementary infinite series for $(1+x)^m$, e^x, $\log(1+x)$, $\sin x$, $\cos x$, $\tan x$."

The Differential and Integral Calculus is to include "simple applications to the properties of curves, to turning values, and to easy mechanical and physical problems." This is the first time we meet with the word "easy"; one could wish for some clearer indication as to its meaning in this case. The Coordinate Geometry opens magnificently by demanding "straightforward applications to the straight line, circle, ellipse, parabola, hyperbola, cycloid, catenary, logarithmic spiral, and other curves of common occurrence." But we are informed later that "a systematic knowledge of conics is not required." The programme concludes with "the elementary statics of liquids and gases."

We have quoted to this extent from the syllabus to justify our complaint that it is lacking in that definiteness, which teachers have a right to expect from their dictators, but more especially to show that the qualifying negatives, with which the instructions bristle, and the extensive range of subjects demanded from boys, who cannot specialise in mathematics, indicate that the framers of this syllabus expect the subject to be dealt with on a new and original plan. We are more anxious to find out what is actually required than desirous to criticise the language employed; we therefore turn to the sole supplementary source at present available, the papers set in November, 1905.

The questions set involve little else but numerical calculations and graphic work. There is, however, one notable exception, which is the following. "P, Q, R are the mid points of the sides of a triangle. Construct the triangle and prove the theorem on which your construction depends." The last part of this question seems to be a sort of a riddle without an

answer, but we are glad to note such an explicit admission of the existence of theorems.

In the two papers set for Math. I. the subjects are represented by the following numbers of questions: Arithmetic, 2; Geom. Drawing, $3\frac{1}{2}$; Theoretical Geometry, $1\frac{1}{2}$; Algebra, 0; Graphs, 2; Theoretical Trigonometry, 0; Numerical solution of Triangles, 3; Statics (Graphical), $2\frac{1}{2}$; Statics (Theoretical), $1\frac{1}{2}$; Dynamics, 0; Mensuration, 2. From this analysis of the papers it is abundantly evident that the examiners do not require a sound theoretical knowledge of any subject, but only the ability to apply special formulae and methods to the numerical solution of a certain class of questions which they regard as of practical importance. Questions are also set involving measurements and weighing; these are dignified by the title of "Practical Mathematics." On this point it is useless to enlarge; the great merit of mathematics as an educational subject is to teach deductive reasoning; whether the educative effect of these experiments is proportional to the time spent on them is open to question, but as a part of mathematics they seem somewhat out of place.

The papers set for Math. II. further confirm the impression hitherto made, and we are forced to the reluctant conclusion that the candidates are expected to be in possession of a few knoblets of knowledge which will enable them to solve questions of a strictly practical and "useful" kind.

If this is really the aim of the army authorities, if they desire mechanical manipulation of formulae rather than intelligence; in short if they prefer the less to the greater they will assuredly reap what they sow. F. E. ROBINSON.

ON A SUPPOSED SOLUTION OF THE "FOUR-COLOUR PROBLEM."

IT appears from the recently published life of Archbishop Temple that he is supposed to have found a proof of the proposition that any map can be coloured with four colours, no two adjacent segments of it being coloured alike. The writer had heard of this before the publication of the Life, but had not succeeded in finding what the proof was, and was under the impression that it had never been published, especially as it was not mentioned in Mr. Rouse Ball's book. As stated in the Life the supposed solution was published in the *Journal of Education*, June 1, 1889. It turns out to be what one has learned from experience (and the writer owns that he has spent much time on the subject) to expect from any theory which looks like a short solution of this extremely elusive problem.

In beginning the investigation one comes at once upon simple principles which promise to contain the key, *e.g.* that the belt of segments surrounding a given segment only requires at most three colours beside that of the given segment: but such principles are only immediately applicable to a figure constructed on some given uniform plan, and in the attempt to extend them to every mode of construction all kinds of complications and difficulties arise.

Temple's theorem is just of this character. His investigation stops before the real difficulties begin, and it may be suspected that many of those who have tried the puzzle have got as far as he has, either by arriving at the self-same theorem, or something analogous to it, and similarly related to the whole problem.

In fact Temple's principle occurred to the writer in a simple self-evident form at an early stage in his own attempts, but he soon satisfied himself that it was no general solution.

The theorem Temple proves is this:—

"There can be four figures on the same plane, no one of which shall have a common part with any other, and every one shall be partially conterminous with every other, but there cannot be more than four."

The essential part of this had occurred to the writer in the following form:—

"If three segments of a map are adjacent each to each, then if a fourth segment is adjacent to each of them, one at least of the first three must be completely enclosed."

The theorem in either form is only a complete solution for a figure constructed as follows: make three segments adjacent each to each; add another adjacent to each of the first three, and leaving therefore at most two unenclosed. Treat the external three segments which may thus result in the same way by adding another adjacent to them. If only one of the original three is unenclosed, there will be two external segments; add another adjacent to both, and treat these three like the first three. Similarly if all of the first three are enclosed.

The theorem could only apply immediately to every figure under certain presuppositions which are certainly not true of every figure.

(i) It would have to be presupposed that if three segments necessarily have three different colours they must be adjacent each to each. But this raises the question whether three segments not thus adjacent might necessarily have three different colours because of their relation to other parts of the figure. And it is easy to shew that this may be the case. It would be necessary therefore to prove that three such segments can only arise in such a way that either (1) not more than one other segment can be adjacent to all three, or (2) that if there could be

two others adjacent to all three, these last two cannot be adjacent to one another. This would be a new and difficult investigation and require new considerations.

(ii) It would have to be presupposed also that if a segment D_1 is adjacent to three segments A_1, B_1, C_1, which must have three different colours A, B and C, and so must itself have a fourth colour D, any other segment X_1 adjacent to D_1 could only be determined to a colour different to A, B and C by being adjacent to the same set A_1, B_1 and C_1. But it would have to be considered whether there might not be another set of three segments which necessarily have the same three colours A, B and C; and it can be shewn again that this is possible.

If then we call such a set A_2, B_2, C_2, we should have to shew that it could only arise in such a way that either (1) no other segment could be adjacent to every member of it, or no other than D_1; or (2) that if a segment X_1 could be adjacent to A_2, B_2 and C_2 it could not be adjacent to D_1. (Other combinations also are possible.) These again are new and difficult investigations requiring, as before, new considerations.

Further, if a map is sufficiently complicated, it will often be found on trial, that if we colour it continuously starting from a segment taken at random, and never introduce a new colour till it is necessary, we shall nevertheless, in closing upon a part of the figure already coloured, get segments of the same colour adjacent to each other, and an alteration of some of the colouring done will be required.

A really satisfactory solution therefore ought to give a definite rule of procedure by which any map whatever can be coloured as required with four colours only. The supposed solution before us clearly does not give any such rule. J. Cook Wilson.

ILLEGITIMATE DIFFERENTIATION.

It has been my experience, and probably that of most teachers, that even advanced students who ought to know better cannot differentiate the simplest algebraic and trigonometric functions without indulging in an extravagant luxuriance of infinite series and making a number of illegitimate assumptions.

In a note in the *Mathematical Gazette* for May, Prof. Bromwich calls attention to the objectionable use of infinite series in differentiating powers. I find that students, who have gone through an extended course of higher mathematics, cannot correctly differentiate even such simple expressions as $\sqrt{x}$ or $\sin x$. In the latter case they put $\sin dx = dx$ and $\cos dx = 1$, regardless of the fact that the latter assumption gives rise to a

term which ultimately takes the form $(1-1)\div 0$, to say nothing more.

It is important that beginners, who are going on to courses in engineering or physics, should learn the meaning of a differential coefficient and an integral long before they are introduced to the theory of convergence and divergence, and should not be allowed to use infinite series until they can be carefully told when the use of such series is legitimate. There is no reason why the study of the Calculus should be postponed till after the Binomial Theorem has been learnt for a positive integer, not to mention for a fraction.

I have examined several text-books with the intention of recommending them to engineering students beginning the Calculus, but have found none which supply exactly what is required. Most text-books are so full of $e^{\sin^{-1}x}$ and $\log \tan \frac{1}{2}x$ and similar expressions that the principles of the Calculus itself are quite hidden from notice.

The proper way of beginning the Calculus is by applying it to simple rational algebraic expressions such as $2x^2-3x+4$. The rules for the differential coefficient of a sum and product should then be given, and the latter illustrated by verifying it in the case of a product of two linear factors such as $(3x-1)(2x+5)$.

The differential coefficient of a power then follows at once by repeated applications of the product rule. When the beginner has got so far he can learn to find the slope of the graph of any simple rational algebraic expression, and to work other illustrative examples such as calculating velocity at any instant from the formula for the space described, and by this time he will be familiar with the nature and meaning of a differential coefficient.

(*a*) If desired he can then learn the rule for the differential coefficient of a quotient, which leads to the differentiation of negative integral powers and simple fractions.

When it is necessary to differentiate powers with fractional indices (and up to this stage no knowledge of the theory of fractional indices is needed), the student should be introduced to the rules for the differential coefficient of a function of a function

$$\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=\frac{dy}{dz}\Big/\frac{dx}{dz},$$

and the differential coefficient of $y=x^{p/q}$ obtained by putting $x=z^q$ and $y=z^p$ in this rule.

(*b*) There is no reason to begin (*a*) before introducing the Integral Calculus. The areas of the graphs of simple rational algebraic expressions afford abundant exercises in integration, and the calculation of areas and possibly volumes is interesting to beginners. Theoretically speaking there is no reason why a beginner should not even learn to calculate moments of inertia in

simple cases before he has learnt to differentiate a quotient. Whether this plan would be expedient it is not our object to discuss. At any rate the volumes of pyramids might be found.

Most beginners have very hazy ideas regarding the why or wherefore of the quantity e as a base of logarithms. Now e need not be introduced until its use is made obvious by trying to differentiate with respect to an index, and it is then conveniently introduced in the following manner :—If we try to differentiate $y=a^x$ we obtain by the rule,

$$\frac{dy}{dx}=a^x \times \operatorname*{Lt}_{h=0} \frac{a^h-1}{h}.$$

The slope of the curve $y=a^x$ is thus proportional at any point to the ordinate y, and a rough idea of the above limit can be got by drawing the graph. The limit represents the slope of the graph at $x=0$.

Now let e be a quantity defined by the property that the slope of the graph of the curve $y=e^x$ when $x=0$ is equal to unity; we then have

$$\frac{de^x}{dx}=e^x.$$

To show that the definition leads to a finite value of e, draw the graphs of $y=2^x$ and $y=4^x$. In the former, the slope of the chord joining $x=0$ and $x=1$ is unity, the slope of the chord joining $x=-1$ and $x=0$ is $\frac{1}{2}$, hence the slope of the tangent at $x=0$ is between $\frac{1}{2}$ and 1. In the latter curve by drawing the chord joining $x=-\frac{1}{2}$ and $x=0$, and the chord joining $x=0$ and $x=\frac{1}{2}$, the slope of the tangent at $x=0$ is shown to lie between 1 and 2. There must, therefore, be a value of a between 2 and 4 such that the slope of the tangent of $y=a^x$ at $x=0$ is equal to unity.

The differentiation of a^x now follows by putting $a=e^n$, or $n=\log_e a$, giving

$$\frac{da^x}{dx}=\frac{de^{nx}}{dx}=\frac{de^{nx}}{dnx}\,\frac{dnx}{dx}=ne^{nx}=a^x\log_e a,$$

and from this starting point the whole of the bookwork associated with the differentiation and integration of exponentials and logarithms can be done without once introducing an infinite series.

There are other ways of approaching the subject, and of these, perhaps, some method of defining natural logarithms without reference to a base in the first instance, using the notation lognat x, may be the best. The definition of logarithms by means of a base in the elementary teaching of mathematics is open to many objections. It tends to make the beginner think that

when mathematicians had practically an unlimited choice of convenient simple *commensurable* numbers which they might take as their base, they chose an inconvenient incommensurable number as base, as the schoolboy said "to make it harder for us." The *value* of the discovery of logarithms had little to do with the base, and depended entirely on the property that the numbers discovered enabled arithmetical calculations to be simplified by replacing multiplication by addition. If then logarithms are defined by the multiplication theorem it is seen that when one such set of numbers have been found, every set of equimultiples of these numbers possesses the same property; in other words that different systems of logarithms only differ by a constant factor. How many beginners fully realise this? In the common logarithms the constant factor is chosen to make $\log 10=1$, and the *base* of any system is merely the number whose logarithm is unity.

If we define lognat x as being equal to $\int dx/x$ between 1 and x, in other words to the area of the hyperbola $y=1/x$ measured from $x=1$ it is easily shown that lognat x is a function satisfying the multiplication theorem; probably a more convenient way would be to define lognat a as the limit of $(a^h-1)/h$ so that

$$\text{lognat } a=\frac{1}{a^x}\frac{d}{dx}(a^x),$$

then the rule for the differentiation of a product gives at once lognat $(ab)=$ lognat $a+$ lognat b, and the rule for differentiation of a power gives lognat $a^n=n$ lognat a showing that natural logarithms possess the same properties as ordinary logarithms. The natural logarithm of any number a could then be constructed graphically by measuring the slope of the graph of a^x, and the base e of the natural logarithms thus defined.]

The mistakes that commonly occur in differentiating sines and cosines arise from $\sin(x+h)$ being expanded by the addition formula instead of $\sin(x+h)-\sin x$ being expressed as a product. This mistake can be avoided in an instructive way by obtaining the differential coefficient from the limit of

$$\frac{f(x+h)-f(x-h)}{2h} \text{ instead of } \frac{f(x+h)-f(x)}{h}.$$

I wonder how many people there are who know that the former fraction gives a better approximation to $f'(x)$ than the latter which is commonly used in defining $f'(x)$. It is not necessary to know Taylor's series to see that such is the case, it may be seen geometrically by drawing the corresponding chords and tangents to any simple graph.

The second mistake is made when $\sin h$ is put equal to h or $\sin \frac{1}{2}h$ equal to $\frac{1}{2}h$ instead of $\sin h/h$ being put equal to unity.

To a person with extended mathematical knowledge either mode of expression is equally significant, but to a beginner the statement $\sin h = h$ is hardly likely to convey more meaning than that because $\sin h$ and h are both zero, one is equal to the other, which is incorrect. To understand the distinction the study of the Calculus forms the best preparation, therefore the statement should not be made at the earliest stage of that study.

It is for this and other reasons desirable that the differentiation of sines and cosines should be proved in the first instance independently of circular measure in the form

$$\frac{d}{dx}\sin x = \cos x \times \underset{h=0}{\mathrm{Lt}} \frac{\sin h}{h},$$

so that the differential coefficient of $\sin x$ is recognized to be *proportional* to $\cos x$ whatever be the unit of angular measurement. That the limit is finite is shown by the fact that the graph of $\sin x$ has a finite slope.

The necessity for this treatment is shown by the fact that science student seven in the second or third year of a university course often *cannot give even the correct answer* in differentiating the sine of an angle expressed in degrees.

What has been said about e applies with equal force to the premature introduction of circular measure into elementary courses. Why mathematicians should be so silly as to revel in a unit of angular measurement which makes all angles that are of any use incommensurable, must be a puzzle to anyone who thinks for himself, and must deter many boys from applying common sense to mathematics. Practically the use of circular measure is not obvious till we try to differentiate sines and cosines.

By confining attention in the first instance to rational algebraic functions, Taylor's and Maclaurin's Theorems may be introduced without the use of infinite series, and may be applied to such exercises as finding the result of substituting say $x = 2 + y$ in $x^3 - 3x^2 + 5x + 2$. These might very well be treated simultaneously with the Binomial Theorem for a positive integral index.

It is surprising that the late Mr. R. A. Proctor in his interesting little book did not differentiate a quotient without introducing an infinite series of "higher powers which may be neglected."

In conclusion, I would not introduce a single infinite series into Elementary Calculus until perhaps the very end when it might be safe to put a few of the expansions of the principal functions (including the Binomial Theorem) into the hands of the learner. Infinite series are dangerous tools which in many cases complicate rather than simplify. They ought to be tabooed from the elementary teaching of mathematics, together with the

base e, the radian, and the poundal, not to mention the still worse "engineers' unit of mass"; and any candidate making use of these means of computation, without reasonable and sufficient cause shown, should be liable for each offence to a penalty not exceeding FIVE MARKS. G. H. BRYAN.

THE DISCUSSION OF CERTAIN POWER-SERIES.

THE difficulties which arise in teaching the elementary power series appear to form part of a larger question, namely, the proper course of mathematical education after a sufficient knowledge of Elementary Geometry, Algebra and Trigonometry has been acquired. The question of Geometry is hardly germane to the present discussion, and I will only mention it so far as to say that I believe it is the hope of many teachers of mathematics that boys of 15 or 16 may in the future learn far more of Projective and Descriptive Geometry than is the case at present.

I imagine that the average student of thirty years ago went through the whole of his University course without any doubt as to the validity of proofs which are now generally regarded as inaccurate. At a somewhat later date such a student imbibed these proofs in the earlier part of his education only to find a year or two later that he had been deceived, and he thus acquired an unsettled feeling as to the validity of much of his previous knowledge. At the present time he is, in most cases, faced with proofs of great complexity, without any idea of the underlying principles, and probably ends by more or less consciously cramming them for examination purposes. The question which I wish to raise concerns the nature of these principles and the best method of expounding them.

The modern analysis appears to be based primarily on two concepts, number and function, and I believe that those who are better qualified than myself to discuss this matter would say that no accurate understanding even of the elementary series can be attained without previous knowledge of these concepts. On the other hand, if the scholarship candidates next autumn were compelled to write short essays on "Incommensurable numbers and the methods of operating with them," and "The meaning of the term function with illustrative examples of different types," no one will doubt that the average attempt would be unsatisfactory. I have therefore, in my own teaching, attempted to deal with these questions first, and will now proceed to give the following summary of the course, not as claiming that it is or approximates to a perfect solution, but rather with the hope that it may serve as a basis for discussion of the nature recently invited by the Editor.

Incommensurable numbers are treated in the method of Jordan in his "Cours d'Analyse" but entirely on a numerical basis, and the theory of limits in general follows naturally. Concrete examples include surds, the calculation of logarithms by successive approximation, the length and area of a curve. Meantime a series of curves are drawn by the students from numerical calculation, these being selected so as to show that the singularities discussed later are real and such as may actually occur in functions already considered. They include

$$y=(1+x)^{\frac{1}{x}},\ y=\frac{10^x-1}{x},\ y=x^n(10^{\frac{1}{x}}-1)/(10^{\frac{1}{x}}+1) \text{ for } n=0,\ 1,\ -1,$$

and

$$y=x^n\sin\frac{\pi}{x} \text{ for } n=0,1,\ -1.$$

The general definitions of a function and its continuity are then discussed at some length, of course with graphical illustrations. Curves are drawn at random on squared paper and the values of the range $(\pm h)$ of x for a given maximum variation (ϵ) are plotted, thus introducing the idea of open and closed ranges for x and the uniform continuity of the function in a given range.

The differential coefficient is then considered—with illustrations depending largely on the individual students—and the differential coefficients of simple algebraic and trigonometrical functions are found, that of x^n being deduced from $\underset{t=0}{dt}\dfrac{t^n-1}{t-1}$. Then follow the theorem of mean value and simple properties of the differential coefficient, and *from these* are deduced separately, as easy examples, the inequalities

$$\frac{x^p-1}{p} \gtrless \frac{x^q-1}{q} \text{ and } ma^{m-1}(a-b) \gtrless a^m-b^m \gtrless mb^{m-1}(a-b).$$

The differentiation of a^x is then attempted (a being of course positive), and is found to depend on $\underset{h=0}{\text{Lt}}\dfrac{a^h-1}{h}$. This is proved, by the above inequalities, to be a number lying between $\dfrac{a-1}{a}$ and a, *independent of the manner in which h approaches zero* and therefore only dependent on a. It is therefore a function of a, say $l(a)$, defined uniquely for all positive values of a. The following properties are then deduced: $l(x_1)+l(x_2)=l(x_1x_2)$. $l(x)$ is negative if $x<1$, zero if $x=1$ and positive if $x>1$, always increases as x increases, and its modulus becomes large without limit as $|x|$ becomes large or small without limit. It is (uniformly) continuous for all finite values of $x>0$, always increases as x increases, and has $\dfrac{1}{x}$ for differential coefficient. Hence, if any

value whatever (y) is assigned to $l(x)$, there always exists one and only one (positive) value of x, and as y increases from $-\infty$ to ∞, x increases from 0 to ∞. The equation $y=l(x)$ may therefore be regarded as defining x as a function of y, say $E(y)$, which always increases with y, is never negative, is (uniformly) continuous, and becomes large or small numerically without limit as $|y|$ becomes large without limit in the positive or negative direction. The properties $E(0)=1$, $E(y_1+y_2)=E(y_1)E(y_2)$, $E(y)=[E(1)]^y\equiv e^y$ say, and $\dfrac{d}{dx}[E(x)]=E(x)$ are easily proved, and the result $x=e^y$ leads to $l(x)=\log_e(x)$. The number e is found $\left(\text{by transformation of } \underset{h=0}{\mathrm{Lt}}\dfrac{a^h-1}{h}\right)$ to be equal to $\underset{n=0}{\mathrm{Lt}}(1+n)^{\frac{1}{n}}$. From this result e is calculated to three decimal places.

The calculation of the expansions for the various elementary functions then follows on somewhat similar lines to those indicated by Professor Bromwich, the only notable difference being in the method used for the Binomial theorem.

The remainder of the course on real variables does not seem to require description here. The definite integral is of course treated fully, and it is pointed out that integration is not to be regarded as a process which may or may not be possible, but as an operation which may give rise to new functions, $\int_1^x \dfrac{dx}{x}$ and $\int_0^x \dfrac{dx}{1+x^2}$ being taken as examples.

As regards complex quantities, they are of course fully discussed in the first place, considerable practice being given in the use of the Argand diagram for such equations as $\omega=z^n$. The question of the definition of a function of a complex quantity is touched upon and the difficulty pointed out; the general definition is not given, it being suggested that the more advisable course is to examine first the properties of functions already known or immediately to be discovered. Then follow the differentiation of z^n and a^x (a being complex and x real), the latter leading as before to $\underset{h=0}{\mathrm{Lt}}\dfrac{a^h-1}{h}\equiv L(a)$. By means of the previous results the proof that

$$L(a)=l(|a|)+i\ \arg.\ a$$

is very simple, and properties of $L(a)$ corresponding to those of $l(a)$ are easily deduced, the most important being of course that if $\omega=L(z)$, a band of the ω plane corresponds to one revolution on the z plane, $L(z)$ being therefore many valued. If ω is confined to any one band of its plane, we may now write $z=E(\omega)$, where $E(\omega)$ repeats itself for different bands, and is therefore

periodic. The properties of $E(\omega)$ are now deduced as before, including

$$E(u)=e^u,\ E(iv)=\cos v+i\sin v,$$

$$E(\omega_1+\omega_2)=E(\omega_1)E(\omega_2),\ E(u+iv)=e^u(\cos v+i\sin v),$$

$$\frac{d}{d\omega}\left\{E(\omega)\right\}=E(\omega) \text{ and } z^n=E[nL(z)].$$

The expansions for the various functions then follow on lines similar to those adopted for real variables.

As already indicated, I believe that some such course should precede any general discussion of infinite series, because it is at once simpler and more in harmony with modern developments than the old method of commencing with such series, whether preceded or followed by some account of their general theory. The latter proceeding appears to me calculated rather to crush and deaden the mind than to induce the true mathematical spirit, for the series are not introduced as the development of any general theory, and the student is forced on to them instead of being led up to their consideration. Moreover the nature of such series cannot be fully understood without considerable discussion of their properties, including the theories of uniform convergence and semi-convergent series, and these are certainly not fit subjects for a first course of analysis.

The above is the outline of a course which is intended both for students who wish to extend their mathematical education considerably beyond its limits and for those who intend to specialise in other subjects. Its success is essentially dependent on a large amount of numerical calculation and drawing being performed by the students, and on this understanding I hope that it may be of some use to the latter class as well as the former.

G. St. L. Carson.

REVIEW.

The Continuum as a Type of Order: an Exposition of the Modern Theory. With an Appendix on the Transfinite Numbers. By Edward V. Huntington. (Reprinted from the *Annals of Mathematics* for 1905; Cambridge, Mass., 50 cents.)

Mr. Huntington's first object is to give a systematic elementary account of the modern theory of the "continuum" regarded as a type of order.

Everyone who has had much to do with mathematics uses the conception of "the continuous independent variable," as represented geometrically by a point on a straight line (if the variable be real) or a plane (if it be complex). It was Dedekind and Cantor in 1872, and Cantor in 1882, who first gave us a clear and purely arithmetical account of the system of the real numbers and exactly what one means by saying: "the system of real numbers, or such and such a space, has

continuity." And, in 1895, Cantor showed that the ordinal type (θ) of the series (0 ... 1) of real numbers is characterised by certain *purely ordinal* properties, in which no mention of, *e.g.*, "distance" occurs.

Russell has emphasised and developed the purely ordinal aspect. For him a "continuum" is a series of type θ, and he calls this "Cantor's second definition (1895) of continuity." I do not think Cantor ever called every such series a continuum, and, indeed, most mathematicians would call a series of some, but not all, of the real numbers in (0 ... 1) "discontinuous." Yet Harnack* has shown that such a series may be of type θ. However, Russell's theory is important to mathematics because many parts of function-theory can be shown to have a purely ordinal basis.†

Mr. Huntington explains lucidly, and by making use of much of Russell's work, what we mean by classes, simply-ordered classes or series, progressions (of type ω) and regressions (of type $^*\omega$), "dense" series (of type η), and "continuous" series (of type θ). This part of the work will be found useful, especially as Russell's contributions do not seem to be understood by everyone; while it seems to be a requisite in teaching function-theory by exacter methods to give some clear ideas of the ordinal properties of the number-system. An attempt (not quite, I think, successful) has been made in this direction by Harkness and Morley in their *Introduction to the Theory of Analytic Functions.*

In an Appendix, Mr. Huntington gives an account of the theory of well-ordered series and Cantor's transfinite numbers. Since much of the latter theory is still in the controversial stage (the laboratory stage of mathematics), an attempt at a connected account can, as yet, hardly be satisfactory. I will only mention one point. Mr. Huntington frequently refers to a paper by Dr. E. W. Hobson, which has served the useful purpose of directing attention to a point which, however, Dr. Hobson did not clearly see. This point is the question of the truth of a certain axiom; and Dr. Hobson would seem to doubt this truth, an opinion in which Mr. Huntington apparently seconds him. There is no reason why this alternative should not be adopted, but it must be observed that the conception of the product of an infinity of cardinal numbers (for example) involves this axiom.

This example shows the difficulties of giving an adequate account of a theory whose consequences are not yet at all fully realised.

PHILIP E. B. JOURDAIN.

MATHEMATICAL NOTES.

(Mr. Jourdain's Note on p. 327 should be numbered **201.** [D. 2. a.].)

202. [D. 2. a.]. Mr. Jourdain does not convince me. The order of ideas of which he would enforce the adoption is orthodox and adequate. Can he prove the contrary with regard to the order which starts from Dedekind's "Schnitt." In Mr. Jourdain's reference (end of § 6) I find that

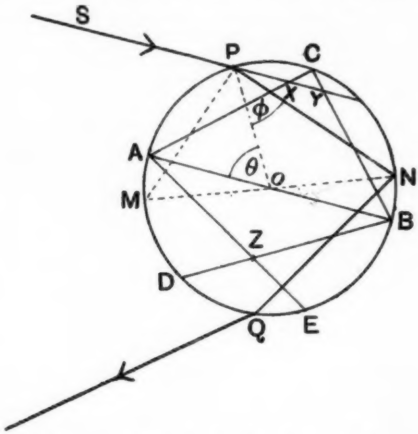
* *Math. Ann.*, Bd. 23 (1884).

† See Russell's *Principles*, pp. 296, 326; and part 2 of my paper in *Crelle's Journal*, Bd. 128 (1905).

Tannery before me has succeeded in annexing to it the Cantor-Weierstrass principle. Granted the "Schnitt," Mr. Jourdain's line 5 is irrelevant.

E. B. E.

203. [T. 3.]. The following is a simple construction for the paths of a series of parallel rays of light of given refrangibility through a uniform transparent sphere, with one internal reflection; as, *e.g.*, the solar rays through a raindrop, in the theory of the Rainbow, in Geometrical Optics.



Suppose an incident ray SP to be in the plane of the paper parallel to the diameter AOB of the sphere, O being its centre; and suppose that the straight line SP produced meets the chords AC, BC in X, Y respectively; C being taken so that $BC : BO$ in the ratio of the index of refraction. Let D, E be the points of trisection of arc AB; and let AE, BD intersect in Z. Then

(1) If M be the point diametrically opposite to N, the point of reflection of the ray, chord $PM = AX$.

(2) If the ray in question be that which suffers minimum deviation in the course of its passage through the drop, $BY = BZ$.

These results are easily proved. (1) enables us readily to construct the paths of a number of parallel rays of given refrangibility, the course of a ray after reflection being symmetrical with that before. Here C is fixed while the position of P varies. By (2), in like manner, we can trace the paths of those rays of various refrangibilities, which undergo minimum deviation, showing (approximately) the directions in which the various colours will be seen in their greatest intensity. Here the position of P varies with that of C. A series of such figures shows more clearly than any lengthy explanation, the way in which the rays passing through different drops give to an eye in a certain position the appearance of the rainbow. The result will be more obvious if the range of values of the index of refraction is somewhat extended.

Proof.

We have by construction $\mu = 2 \sin BAC = 2 \cos ABC$;

$$i.e.\ \sin\theta = 2 \sin\phi \sin BAC = 2 \sin\phi \cos ABC,$$

where θ, ϕ are the angles of incidence and refraction. Thus we have

1. AX, which $=\dfrac{AO\sin\theta}{\sin BAC}$, $=AB\sin\phi$; *i.e.* $\sin\phi=\dfrac{AX}{AB}=\dfrac{PM}{MN}=\sin MNP$;
And supposing PM taken $=AX$ and MN the diameter through M, $\angle MNP=\angle NPO$, which is therefore the proper angle of refraction.

2. The total deviation $=\pi-2x$, where $x=\angle MOA=\angle MOP-\angle AOP=2\phi-\theta$.

Thus, substituting for ϕ above, we have $\sin\theta=2\sin\dfrac{\theta+x}{2}\cos ABC$, and for a minimum deviation

$$\cos\theta=\cos\frac{\theta+x}{2}\cos ABC.$$

Hence $\cos^2 ABC=\cos^2\theta+\frac{1}{4}\sin^2\theta$; or $\sin ABC=\dfrac{\sqrt{3}}{2}\sin\theta$.

Thus BY, which $=\dfrac{AO\sin\theta}{\sin ABC}$, $=\dfrac{AB}{\sqrt{3}}=BZ$, when the deviation is a minimum.

P. J. HEAWOOD.

204. [D. 2. a.] *Elementary Illustration of the Properties of Infinite Series.*

The writer has not met with a detailed discussion on these lines of the series mentioned below and thinks it may be of use to others who find that students have considerable difficulty in grasping the general theory.

1. Consider the family of curves

$$y=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots-\frac{(-x)^n}{n}\equiv S_n \text{ say,}$$

for which
$$\frac{dy}{dx}=\frac{1-(-x)^{n+1}}{1+x}.$$

It is easy to prove that if $x\leqq 1$ all the curves for which $n>p+1$ lie between the pth and $(p+1)$th curve and that the difference of the ordinates of these latter curves can be made small without limit by unlimited increase of p. Further, if $x<1$ the curves all tend to the common gradient $\dfrac{1}{1+x}$ as n increases without limit, but if $x=1$ the gradient is zero if n is even, and unity if n is odd, so that as they cross the line $x=1$ they part into two rapidly diverging families and there is a finite change of gradient (*not ordinate*) as x increases to unity. This latter change occurs more and more suddenly without limit as n increases indefinitely.

2. Next consider the family

$$y=x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots+\frac{x^n}{n}\equiv T_n \text{ say,}$$

for which
$$\frac{dy}{dx}=\frac{1-x^n}{1-x}.$$

If $x<1$, all the curves after the pth lie within a limited distance above it, which distance can be made small without limit by unlimited increase of n, and as n increases indefinitely they tend to the common gradient $\dfrac{1}{1-x}$. As x increases to unity the gradient tends to the limit n and when $x=1$ the distance of the nth curve above the pth can be made large without limit by indefinite increase of n. Hence, for increasing values of n, the curves diverge more and more rapidly from Ox, *but an upper limit can be assigned to their ordinates so long as* $x<1$, this limit of course increasing indefinitely as x approaches unity.

3. Similarly for the family

$$y=\frac{x}{1^2}+\frac{x^2}{2^2}+\ldots+\frac{x^n}{n^2}\equiv U_n \text{ say,}$$

$$\frac{dy}{dx}=1+\frac{x}{2}+\ldots+\frac{x^n}{n}\equiv T_n$$

if $x\leqq 1$ all the curves after the pth lie within a limited distance above it, which distance can be made small without limit by unlimited increase of n. If $x<1$ their gradients tend to the same finite limit (since T_n does so), but if $x=1$ their gradients increase without limit. Hence as n increases without limit the curves approach one another more and more closely without limit and their ordinates tend to a definite finite value up to $x=1$ *inclusive*, but they cross the line $x=1$ with indefinitely increasing gradients.

4. Finally take the family

$$y=x+\frac{x^3}{3}-\frac{x^2}{2}+\frac{x^5}{5}+\frac{x^7}{7}-\frac{x^4}{4}+\ldots\equiv V_n \text{ say,}$$

$$\frac{dy}{dx}=\frac{1-x^{4m-1}}{1-x^2}-x\frac{1-x^{2m-1}}{1-x^2} \quad (n=3m)$$

$$=\frac{1-x^{4m-1}}{1-x^2}-x\frac{1-x^{2m-1}}{1-x^2}+x^{4m} \quad (n=3m+1)$$

$$=\frac{1-x^{4m-1}}{1-x^2}-x\frac{1-x^{2m-1}}{1-x^2}+x^{4m}+x^{4m+2}. \quad (n=3m+2)$$

If $x\leqq 1$ (the two cases now require separate treatment) all the curves after the pth lie within a limited distance of the pth, which distance can be made small without limit by unlimited increase of n. If $x<1$ their gradients tend to the limit $\dfrac{1}{1+x}$, but when $x=1$ they tend to the limits m, $m+1$, $m+2$ in the three cases above. Hence this case differs from those above in that the *ordinates* are limited up to $x=1$ *inclusive*, while the *gradients* increase sharply as x approaches unity. For large values of n the curves approach the line $x=1$ at a gradient nearly equal to $\frac{1}{2}$ and then turn sharply upwards with a large gradient, crossing $x=1$ with ordinates which do *not* tend to a limit near those for which x is slightly <1.

Further, it is easy to prove that

$$\tfrac{1}{6}x^{6n+1}+\tfrac{1}{4}x^{4n+2}>V_{6n}-S_{6n}>\tfrac{1}{8}x^{8n-1}+\tfrac{1}{6}x^{6n-1},$$

so that if $x<1$ V_{6n} and S_{6n} tend to the same limiting value for unlimited increase of n, but if $x=1$

$$\tfrac{5}{12}>V_{6n}-S_{6n}>\tfrac{7}{24},*$$

and in this case the series converge to different values. This illustrates the discontinuity which may occur in the sum of a series and the change in value caused by deranging the terms of a semi-convergent series.

For teaching purposes the argument should of course be illustrated by actual drawing.

G. St. L. Carson.

* $\frac{5}{12}>\frac{1}{2}\log 2>\frac{7}{24}$ as should of course be the case.

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